

Calculus I: Lab Notebook

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Introduction to the *SageMath* Laboratories

The Mathematics Department has developed a set of computer laboratory assignments to enhance the graphical and numerical understanding of concepts throughout Calculus I. The lab assignments use *SageMath*, an open-source computer algebra system, to guide students through a variety of experiments that explore the ideas of Calculus. In particular, the use of *SageMath* will also allow us to tackle some applications that would have otherwise been out of reach for technical reasons. Moreover, by preparing and organizing their lab reports, students will learn to consolidate mathematical ideas and will improve their mathematical communication skills.

To simplify access to *SageMath* and the lab assignments, the Mathematics Department maintains a dedicated WordPress site:

<https://wp.towson.edu/math273/>

This website hosts online versions of the labs, including the basic *SageMath* functionality required to complete the assignments; it contains also some additional *SageMath* resources.

The labs are mostly self-guided: while the instructor may provide a brief overview or an explanation of a certain point, students are expected to complete the lab mostly on their own. The online version of each lab consists of several screens that mix instructions, the statements of questions, and online interface to run *SageMath* commands.¹ Students must read through the instructions and then use the *SageMath* interface to perform the calculations required to answer the questions.

For each lab, students shall turn in a **physical**, filled lab report for grading. Blank lab reports will be made available as PDF files by the instructor. Each lab report will include a list of questions, based on those in the online assignments, but with some adjustments and with some section-specific content. Students will need to use the *SageMath* interface in the online version of the lab to answer the questions and will then write their answers in the lab report. Some questions may require also the submission of printouts of work completed through the *SageMath* interface.

Moreover, students are required to keep electronic notes of their lab work in this lab notebook. The PDF version of this notebook contains a complete record of the instructions for the labs, including fillable blank fields where students can save any *SageMath* code they used online. As they work through the examples, students are expected to fill those blank fields for future reference. After each lab, the lab notebook provides also a field for notes, where the student can record any information related to the lab, such as: hints or instructions given in class for completing the lab report, reminders to self how to resolve certain issues, etc. Students shall turn a PDF file of their lab notebook at the end of the semester to be graded for completion.

¹To help students prepare for labs by pre-reading the instructions, static versions of most of the online content is included also in this manual.

To the Student

The *SageMath* labs are an indispensable part of the course. The main objective of the labs is for you to engage with the concepts in a hands-on way, where you actively experiment with the meaning and consequences of the core ideas of the course. Their goal is to help you learn how and when to use the equations and techniques that you will see in class.

To prepare for each lab, you should:

- Make sure you have a copy of the blank electronic lab report. Your instructor will provide this to you in advance as a fillable PDF file. Note that there are multiple versions of the reports. Thus, it is important to **use the lab report file provided by your instructor**.
- Review your class notes on the material referenced in the lab. To do this efficiently, skim through the version in this lab notebook² and look for the key terms and results referenced in it. For example, Lab #2 is about limits of functions and seems to be designed to build up your intuition about the concept of a limit. Thus, you will want to review your class notes on the definition of a limit.

As you work through each lab assignment, you will produce two separate outputs:

- You will save some of the code for the sample exercises (from the online version of the lab) in your lab notebook. You may also have to use the lab notebook to save answers to some of the online exercises that are not part of your lab report. As the semester progresses, you will fill up the electronic version of your lab notebook, and after the last lab you will submit the complete lab notebook for grading (consult the syllabus for details).
- You will answer the questions in the lab report. The exercises in the lab report will be based on exercises in the online version of the lab, but the examples will often be different; the lab report versions may also ask additional questions or give specific instructions. Typically, you will use the *SageMath* interface in the online version to perform a task before you answer a question in the report. You will encounter three types of questions that will each interact slightly differently with their online versions:
 - **questions that ask you to type in the result of a calculation:** for each of those, the electronic version of the lab report will provide a fillable field to type your answer. You will need to use *SageMath* to perform a calculation and then will type your answer in that field and save it.
 - **questions that ask for an answer** based on work you completed: for each of those, the electronic version of the lab will provide a larger blank box to type your answer.

²Alternatively, you may use the online version of the lab on the [WordPress site](#).

In general, such questions will not require you to perform any additional work on-line, but instead you will analyze already completed work. Again, you must type your answer in the provided box. Make sure to use complete sentences and proper spelling, grammar and mathematical notation: those elements will be part of your grade on such questions!

- **questions that ask for a printout of the work you did online:** for those, you will need to produce a single sheet of paper with your name that displays the required *SageMath* code and its output. The online versions of such questions are formatted so that each goes on a separate sheet of paper when printed through a web browser. Thus, for each question that asks you to “attach a printout,” you should use the **Print** option of your browser to locate that sheet and print only it. See the screenshot below for a sample how to do this in Google Chrome.

Once you have answered all the questions in your lab report, you must submit a **physical** copy of your completed report for grading; unless your instructor tells you differently, it will be due the next class day. The complete report will consist of a printed version of the PDF file with all the answers (to the first two types of questions) **typed** in their respective fields and with all the required printouts (for the third type of questions) attached—each on a single page. For full credit, your lab reports must follow the above instructions and all pages must be properly arranged and stapled.

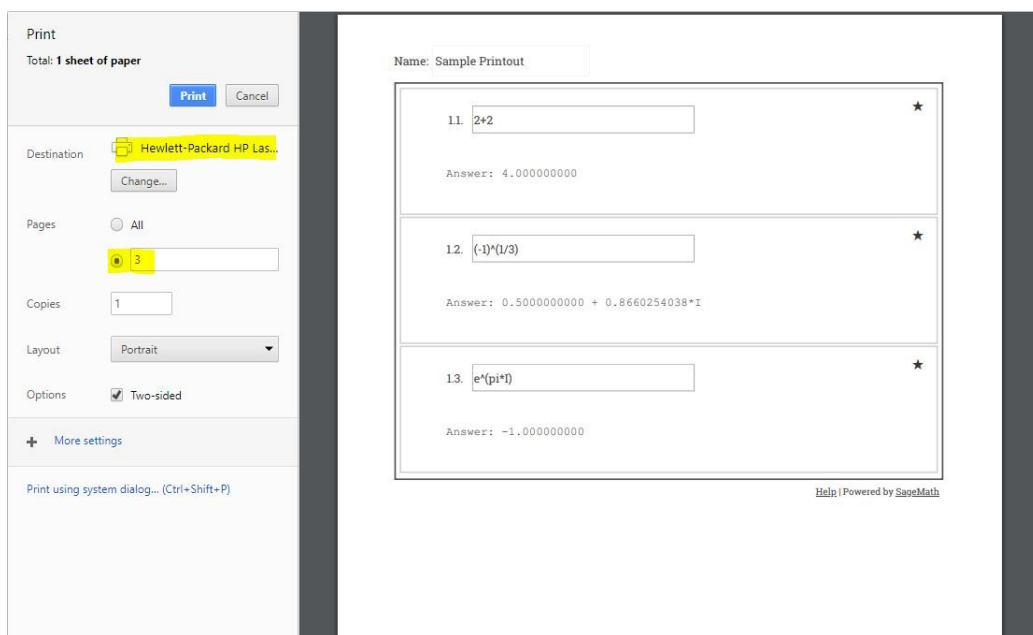


FIGURE 0.1. Preparing to print *SageMath* output on page 3 using Google Chrome. The name of the printer and the page to be printed are highlighted. The printer will print only the page currently displayed. If you change the highlighted printer to the “Save as PDF” option (or “Print to PDF” for other browsers), you may produce also a single-page PDF file of your output.

LAB 1

Review of Functions and Introduction to *SageMath*

FUNCTIONS WITH NAMES OTHER THAN f AND g

Function names do not have to be single letters (like f or g). In fact, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\log(x)$, and $\ln(x)$ are all examples of functions whose names are longer than a single letter. Many of these common functions are built into mathematical software such as *Mathematica* and *SageMath*. Here are some of the elementary built-in functions that you will be using often. When you use the function in a *SageMath* computation cell, it should be entered exactly as it appears in the second column.

Basic Function	<i>SageMath</i> Syntax
$ x $	<code>abs(x)</code>
\sqrt{x}	<code>sqrt(x)</code>
$e^x, \exp(x)$	<code>exp(x)</code> or <code>e^x</code>
$\ln(x) = \log_e(x)$	<code>log(x)</code>
$\log(x) = \log_{10}(x)$	<code>log(x, 10)</code>
$\log_b(x)$	<code>log(x, b)</code>
$\arcsin(x) = \sin^{-1}(x)$	<code>arcsin(x)</code> or <code>asin(x)</code>
$\arccos(x) = \cos^{-1}(x)$	<code>arccos(x)</code> or <code>acos(x)</code>
$\arctan(x) = \tan^{-1}(x)$	<code>arctan(x)</code> or <code>atan(x)</code>

SageMath also predefines some standard constants, including:

$$\begin{array}{lll} \pi = 3.14159\dots & \text{as} & \text{pi;} \\ e = 2.71828\dots & \text{as} & \text{e;} \\ i = \sqrt{-1} & \text{as} & \text{I.} \end{array}$$

FUNCTION NOTATION IS NOT MULTIPLICATION

SageMath uses the symbol $*$ to identify multiplication. It is important to remember that in contrast to regular mathematical notation the multiplication symbol $*$ is **not** optional in *SageMath*: that is, anytime you want to multiply two quantities you must use the symbol $*$. Thus, to enter the expression $5x$, you would write `5*x`.

While this feature of *SageMath* can feel mildly annoying at first, there is one context where it is rather helpful. Oftentimes, the use of parentheses in function notation makes people think that they are multiplying. However, we must think about whether the symbols represent function names or variables. For example, if f is the name of a function and x is a variable, then $f(x)$ is function notation that represents the output to the function f when the input is x ; in this case,

$f(x) \neq f \cdot x$. On the other hand, the parentheses in the expression $x(x)$ do indicate multiplication; in this case, $x(x) = x \cdot x = x^2$. *SageMath*'s convention regarding multiplication means that this sort of ambiguity **cannot** occur in *SageMath*: in the above situation, $f(x)$ represents $f(x)$ in *SageMath*, and $x*x$ (or $x*(x)$, or x^2) represents x^2 , but $x(x)$ and $f*(x)$ result in error messages.

FUNCTION PLOTS IN *SageMath*

Using *SageMath*, we can easily plot functions over specified intervals of the domain. For example, to graph the function $f(x) = x^2 + 2x + 7$ for $-5 \leq x \leq 5$, we enter the following command:

```
plot(x^2+2*x+7, -5,5)
```

We can also graph multiple functions at once. For example, suppose we want to graph the curves $y = f(x)$ and $y = f(x) + 10$, both over the same range for x . To do so, it is convenient to first define the function $f(x) = x^2 + 2x + 7$ and then to have *SageMath* plot both $f(x)$ and $f(x) + 10$ on the interval $[-5, 5]$. This is exactly what the following two lines of code accomplish: the first defines a new function named f ; and the second plots $f(x)$ and $f(x) + 10$ on the interval $[-5, 5]$. Note that when we want to plot more than one function simultaneously, we list those functions within a pair of parentheses, separated by commas: e.g., $(f(x), f(x)+10)$ in the code below.

```
f(x) = x^2+2*x+7
plot((f(x), f(x)+10), -5,5)
```

We can specify the color of each plot as follows:

```
f(x) = x^2+2*x+7
plot((f(x), f(x)+10), -5,5, color=('red', 'black'))
```

Exercises

Exercise 1.1. Type in the *SageMath* code that calculates each of the listed expressions.

$\log_2(1024)$ Code:

$\sin(\pi)$ Code:

$\arctan(0.75)$ Code:

$|-2.3|$ Code:

$\sqrt[3]{-1}$ Code:

$\sqrt{7}$ Code:

Exercise 1.2. The table below lists three mathematical expressions and incorrect *SageMath* code that fails to evaluate those expressions. Fix the given *SageMath* code so that it evaluates the respective expression correctly and type the corrected code in the last column of the table. You may use the *SageMath* cell in the online version of the lab to test your code.

Expression	Incorrect SageMath Code	Corrected SageMath Code
$\log_3(\pi)$	<code>log[pi, 3]</code>	
$\cos(1.2)$	<code>cosine(1.2)</code>	
$e^{ 2\sqrt{2}-3 }$	<code>Exp^ 2sqrt(2)-3 </code>	

Exercise 1.3. Type in the *SageMath* code that calculates each of the listed expressions.

$\sin(\pi^2)$ Code:

$\pi^2(\sin(0.5\pi))$ Code:

$\sqrt{6e^3}$ Code:

$(2e)\sqrt{e}$ Code:

$\sin(\cos(\tan(0.3\pi)))$ Code:

Exercise 1.4. In the blank space below:

- Write a line of *SageMath* code that defines the function $g(x) = \ln(4 - 3 \sin(x))$.
- Write a line of *SageMath* code that plots the curves $y = g(x)$ and $y = g(x) + 6$ on the interval $[-3, 4]$, the first in red and the second in green.
- Write a sentence that describes the effect of adding 10 to the function's output (that is, answer Question 4 in the online version of the lab).

Exercise 1.6. In the blank space below:

- (a) Define the function $f(x) = 2^{3 \sin(3x)-1}$ and evaluate the slope of the line through the points $(0, f(0))$ and $(1, f(1))$.
- (b) Draw a combined plot over the interval $[-2, 2]$ of the graph $y = f(x)$ and the line through the points $(0, f(0))$ and $(1, f(1))$.
- (c) Define a function $m = m(b)$, $b \neq 0$, so that $m(b)$ equals the slope of the line through the points $(0, f(0))$ and $(b, f(b))$.

Notes on Lab 1.

LAB 2

Limits

INFORMAL APPROACH TO LIMITS

When evaluating the limit of a function at c , you are determining the value of the function you would expect at $x = c$ based on the values it has when x is **near** c . The value of the function **at** $x = c$ is not needed to make this determination. In other words, the limit can be evaluated even if the vertical line $x = c$ is covered up to hide the function's output at c . The limit is, in this sense, a prediction at $x = c$ based on the trend of nearby function values. To be precise with this prediction, it is necessary to zoom in on the graph at the point of interest.

To explore this idea, several of the exercises below use animated *SageMath* plots, where a vertical line has been covered up, and the value of the function at that point is unknown. A slider bar allows you to zoom in and out on the part of the graph of a function near the covered point. You can vary the zoom level between 1 and 40.

EVALUATING LIMITS IN *SageMath*

SageMath has the ability to compute the limits of certain functions. For example, to evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x - 1}$$

you would type

```
limit((x^2-2*x+1)/(x-1), x=1)
```

You can also graph the function:

```
plot( (x^2-2*x+1)/(x-1), -1,4)
```

(Recall that the numbers -1 and 4 give the range of x -values to appear in the plot.)

Sometimes, you will want to put together several *SageMath* commands in an evaluation box and see the outputs of **several** of them (as opposed to just the **last** one). This can be accomplished by wrapping the computation whose output we want to see in a `print()` command. To see this in action, try typing the commands below in a *SageMath* evaluation window:

```
print( limit( (x^2-2*x+1)/(x-1), x=1) )
plot( (x^2-2*x+1)/(x-1), -1,4)
```

Exercises

Exercise 2.2. If the values of the function f at inputs near c have a trend so that you can make a reasonable prediction for the value at $x = c$, no matter what the zoom level is, then what can you

conclude? In the blank space below, list the letter(s) of the correct conclusions.

- A** The limit $\lim_{x \rightarrow c} f(x)$ exists.
- B** The graph of f , zoomed in really close to $x = c$, must appear linear.
- C** The trend of values on the left side of $x = c$, and the trend on the right side, give the same prediction.
- D** The prediction $\lim_{x \rightarrow c} f(x)$ must be the same as the actual value, $f(c)$.

Exercise 2.5. For the function $f(x) = \frac{2x - 4}{\ln(x/2)}$, use the above commands to evaluate $\lim_{x \rightarrow 2} f(x)$ and to graph this function over the range $1 \leq x \leq 3$. Write the *SageMath* code you used in the blank space below.

Notes on Lab 2.

LAB 3

Limits II

This lab considers the topic of limits again, but in a more mathematically rigorous manner. In the previous lab, you thought of the statement

$$\lim_{x \rightarrow c} f(x) = L$$

as a prediction of a y -value based on the trend of the values $f(x)$ for inputs x near c . The goal here is to make the idea of “trend of values” more precise. We want those values $f(x)$ to “stay close” to L when we “zoom near enough” to the point $x = c$. To gain a deeper understanding of what this means, we need to quantify the notions “near enough” and “stay close.”

ZOOMING 2.0

To begin, we take a more in-depth look at the zooming process we used in Lab #2. Recall that when we zoom in, we choose a zooming window (a rectangular piece of the plot), look at the graph just in that window, and increase the resolution to look more closely. This window has a rightmost x -value, x_{\max} . If the window is centered at $x = c$, then the rightmost x -value minus c equals half the width of the window—call that number δ :

$$\delta = x_{\max} - c.$$

We use the number δ to control the depth of zooming—the smaller δ , the deeper the zoom (or the larger the zoom level).

Once we have set the width of the window around c (as determined by δ), the graph $y = f(x)$ will typically have a largest y -value and a smallest y -value over the chosen range for x —call them y_{\max} and y_{\min} .¹ In the animated *SageMath* plots, in the online versions of this lab and the last, the height of the zooming windows is chosen so that the top edge of the rectangle is set to y_{\max} and the bottom edge is set to y_{\min} . A sample of this is displayed on Figure 3.1.

Our exploration of the claim that $\lim_{x \rightarrow c} f(x) = L$ will be based on the difference $y_{\max} - y_{\min}$. Note that if all the values $f(x)$ on the zoomed graph are close to L , so are y_{\max} and y_{\min} , and the difference $y_{\max} - y_{\min}$ **must be small**. The first three exercises develop this idea: You are given a small number $\tau > 0$ (such as 0.01, 10^{-3} , or 0.00012)—a threshold below which we consider $y_{\max} - y_{\min}$ to be small. Then you want to find a δ such that in the zooming window of half-width δ satisfies

$$y_{\max} - y_{\min} < \tau.$$

This is a quantitative version of the statement: “the values $f(x)$ stay close to each other when we zoom near enough to $x = c$.”

¹This may fail— y_{\max} or y_{\min} (or both) may not exist, but then the problem we are trying to solve is, in fact, easier. For now, let us focus on the scenario where both y_{\max} and y_{\min} do exist.

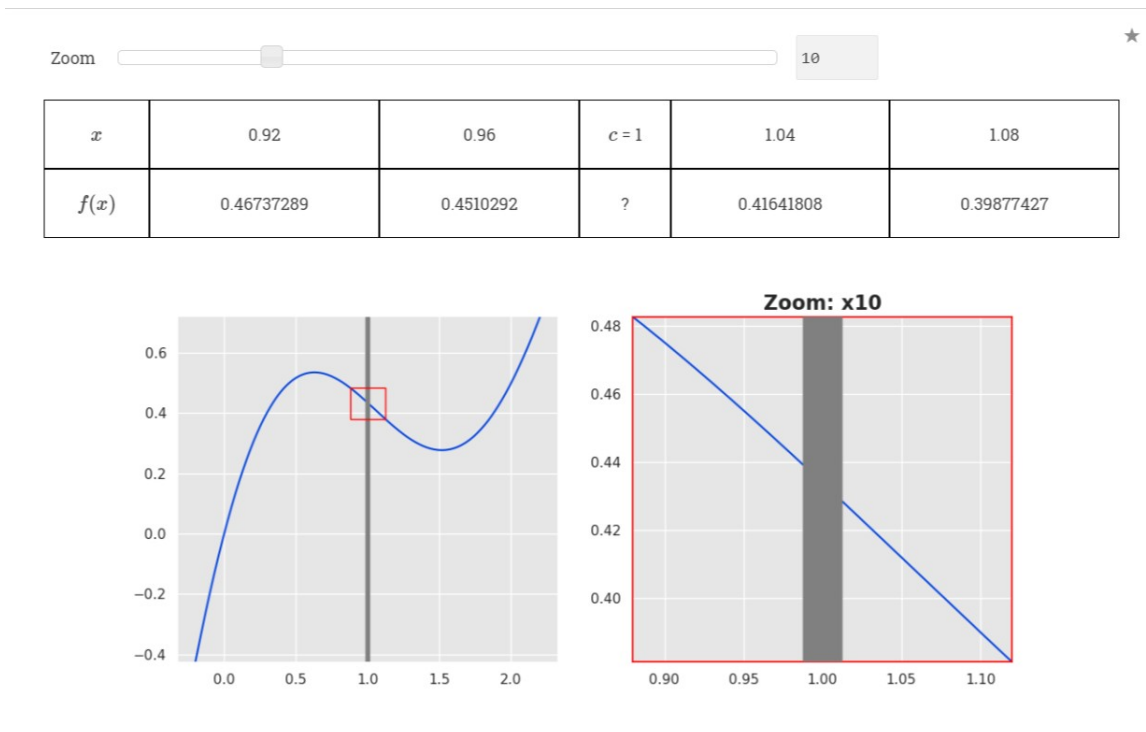


FIGURE 3.1. A sample zoom near $c = 1$: $x_{\max} \approx 1.12$, $y_{\max} \approx 0.48$, $y_{\min} \approx 0.38$.

FORMAL APPROACH TO LIMITS

As explained in the first half of the lab, if we expect that $\lim_{x \rightarrow c} f(x) = L$, then we want to be able to make $y_{\max} - y_{\min}$ less than **any** pre-determined threshold $\tau > 0$, no matter how small, by setting δ (the half-width of the window) small enough. Recall the formal definition of limit from §2.5 of the textbook:

DEFINITION. Let $f(x)$ be defined for all $x \neq c$ over an open interval (a, b) containing c . Let L be a real number. Then

$$\lim_{x \rightarrow c} f(x) = L$$

means that: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that: whenever $c - \delta < x < c$ or $c < x < c + \delta$, one has $L - \varepsilon < f(x) < L + \varepsilon$.

This is another way to say what we earlier expressed in terms of the difference $y_{\max} - y_{\min}$: in this definition, 2ε plays the role of the threshold τ in our earlier discussion, since $y_{\max} - y_{\min}$ will be less than 2ε . When checking this definition, we want to answer the following question: Is it possible to select δ in terms of ε so that the values $f(x)$ will **always** be between $L - \varepsilon$ and $L + \varepsilon$ when the input x falls within the zooming window—that is, when $c - \delta < x < c + \delta$? Is this possible for **all** $\varepsilon > 0$?

The last few exercises in this lab investigate the above question. They deploy a slightly different zooming *SageMath* demo that replaces the zoom level with a choice of δ , though not a direct one. Instead, we get to select an ε in the range $0.005 \leq \varepsilon \leq 1$ and then to select $\delta = k\varepsilon$ for any choice of k in the range $0.1 \leq k \leq 4$. The demo also allows us to make a choice for the value L of the limit,

and then sets the zooming window to the rectangle defined by the inequalities

$$c - \delta \leq x \leq c + \delta, \quad L - \varepsilon \leq y \leq L + \varepsilon.$$

Some additional information is also displayed on the plots: the choices of ε and δ and the values of y_{\min} and y_{\max} for the chosen zoom level are displayed on the large scale plot of $y = f(x)$, and the horizontal line $y = L$ is drawn on the zoomed-in plot of $y = f(x)$.

Exercises

Exercise 3.1. Use the *SageMath* cell in the online version of the lab with the function F1 and zoom level set to 20 to answer the questions below.

- (a) What is the rightmost x -value in the window? Use this to compute δ and record the value below.
- (b) What are the maximum and minimum y -values on the graph in the window? Use this to compute $y_{\max} - y_{\min}$ and record the value below.

$$\delta =$$

$$y_{\max} - y_{\min} =$$

Exercise 3.4. Use the *SageMath* cell in the online version of the lab.

- (a) Set $L = 1$ and $\varepsilon = 0.25$ and experiment with the value of k to find a δ for which at least one of the inequalities

$$y_{\max} < L + \varepsilon \quad \text{and} \quad y_{\min} > L - \varepsilon \quad (*)$$

fails. Record the values you found.

$$\delta =$$

$$y_{\max} =$$

$$y_{\min} =$$

- (b) Still with $L = 1$ and $\varepsilon = 0.25$, experiment with the value of k to find a different δ for which both inequalities (*) hold. Record below the values you found.

$$\delta =$$

$$y_{\max} =$$

$$y_{\min} =$$

(c) Consider the results of parts (a) and (b). What changed in the relationship between the graph $y = f(x)$ and the viewing window that enabled us to satisfy inequalities (*) in (b)?

Notes on Lab 3.

LAB 4

Functions and Their Derivatives

In this lab you will practice distinguishing between applications of the function and applications of the derivative. Some questions will require you to know a little more about *SageMath*.

USING *SageMath* TO CALCULATE DERIVATIVES AND TO SOLVE EQUATIONS

SageMath can evaluate functions by entering two lines of code. The first line defines the function and the second line specifies the output to evaluate. For example, to evaluate $f(3)$ when $f(x) = 4x^2$ enter:

```
f(x)=4*x^2
f(3)
```

SageMath can also compute derivatives and evaluate these derivatives. For example, to determine the function rule for f' when $f(x) = 4x^2$ you may enter the following:

```
f(x)=4*x^2
df(x)=diff(f,x)
df(x)
```

And a slight modification of the last *SageMath* code gives the value of the derivative at $x = 3$:

```
f(x)=4*x^2
df(x)=diff(f,x)
df(3)
```

Finally, you can use *SageMath* to solve equations. In *SageMath* (and many other software packages), definitions are often made with an equal sign (`'='`). For example, when you type $z(x) = x^2 + 1$ into *SageMath*, this **assigns** z to be a function of x after that line.

In an equation, the equal sign has a different meaning; you are not assigning the left-hand side to be the right-hand side for future reference, but are interested in which values of variables make the sides equal. For this reason, *SageMath* needs to use something besides `'='`. The developers of *SageMath* have chosen the double equal sign `'=='` to denote an equation. The online version of this lab contains a *SageMath* computation cell that you can use to find the real roots of many equations: it contains a cell labelled “Equation to solve” and by default displays the equation $2x = 7$ and its solution. If you replace the default equation by the valid *SageMath* code for another equation (say, $4x^2 = 2$), you can solve that other equation. (You will learn how to use the *SageMath* command `find_root()` to solve equations in a later lab.) You are to use those *SageMath* cells to answer the questions in Exercise 4.1 below.

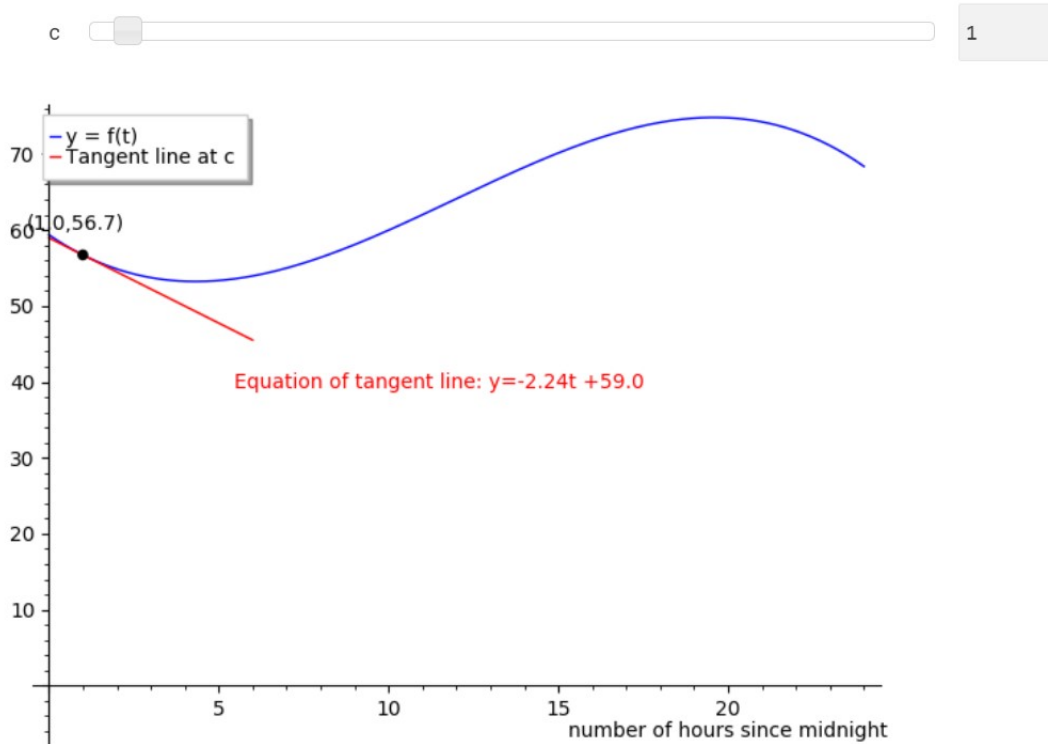


FIGURE 4.1. A sample plot of the tangent line at $t = 1$.

UNDERSTANDING FUNCTIONS AND THEIR DERIVATIVES GRAPHICALLY

Recall that in lecture we studied the secant lines through a fixed point $P(c, f(c))$ on the graph $y = f(x)$ and a moving point $Q(b, f(b))$, where $b \neq c$. As we discussed in lecture, the limit of the slopes of these secant lines, as $b \rightarrow c$, is the slope of the tangent line at $x = c$. As this limit is also equal to the derivative $f'(c)$, the **tangent line** at $x = c$ is the line which:

- (i) passes through the point $(c, f(c))$;
- (ii) has slope $f'(c)$.

The online version of the lab contains a couple of *SageMath* demonstrations. Each of them displays the graph of a function (in blue) and its tangent line (in red) at a point $x = c$ that you can control via a slider bar (see the screenshot in Figure 4.1). You are to use those *SageMath* demonstrations to answer the questions in the exercises in your lab report.

Exercises

Exercise 4.1. In a medical experiment, the body weight in grams of a baby rat after t days was $f(t) = 4.96 + 4.8t + 0.17t^2 - 0.0048t^3$. Use the *SageMath* computation cells below Exercise 1 in the online version of the lab to answer the questions below. Only use solutions that make sense in the context of the experiment. Include brief explanations with your answers.

- (a) Approximately how much did the rat weigh after 7 days?

- (b) Approximately when did the rat's weight reach 27 grams?

- (c) Explain how the number of solutions you found in part (b) makes sense given the context.

- (d) Approximately how fast was the rat gaining weight after 4 days?

Exercise 4.2. Exercise 2 in the online version of this lab contains an animated plot of a function $y = f(t)$ that represents the temperature inside a house (in degrees Fahrenheit) as a function of t , the number of hours elapsed since midnight. The animation allows you to select a time $t = c$ and to display the tangent line to the graph at $t = c$ (see Figure 4.1 for a screenshot of the tangent line at $t = 1$). Use that demonstration to answer the questions below. Include brief explanations with your answers.

- (a) What is the temperature of the house at 6 am?

- (b) How fast is the temperature of the house changing at 11 am?

(c) By how much did the temperature change between 6 am and 2 pm?

(d) What was the average rate at which the temperature changed from 6 am to 2 pm?

Notes on Lab 4.

LAB 5

Implicit Differentiation

IMPLICITLY DEFINED CURVES

Consider the equation

$$6x + 3y = 2.$$

We can solve this equation to get y explicitly as a function of x —namely, $y = \frac{2}{3} - 2x$. On the other hand, the equation

$$y^4 - y - \sin(2\pi x) = x^3 + x$$

is much more difficult to solve for y . (Certainly, the math required to solve it goes beyond a basic calculus course.) Even though we cannot express the relation represented by this equation as an *explicit* function $y = y(x)$, the equation still determines a curve in the plane—you can see its plot in Figure 5.1.

This curve is made of points with coordinates (x_0, y_0) which have the property that setting $x = x_0$ and $y = y_0$ in the equation gives left- and right-hand sides which are, in fact, equal. For example, the curve contains the point $(x_0, y_0) = (0, 1)$, since

$$1^4 - 1 - \sin(2\pi * 0) = 0^3 + 0.$$

The curve shown in Figure 5.1 is exactly the set of all such points (x_0, y_0) , where $-2 \leq x_0 \leq 10$ and $-5 \leq y_0 \leq 5$.

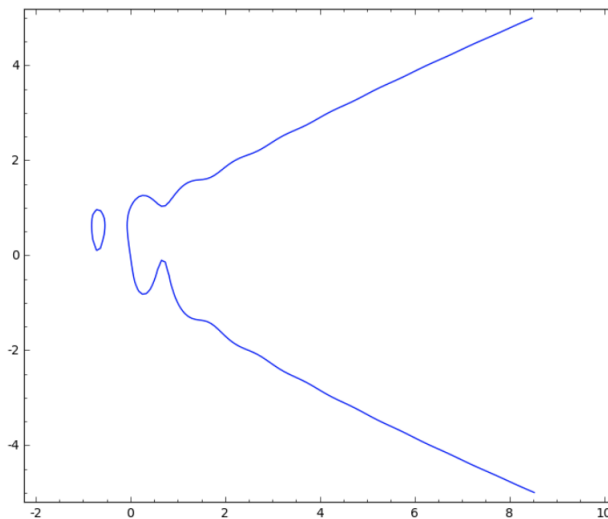


FIGURE 5.1. The curve $y^4 - y - \sin(2\pi x) = x^3 + x$.

SLOPES OF TANGENT LINES ON A CURVE

When we graph a differentiable function, we have a tangent line at a point on that graph. Similarly, there will be tangent lines at points on implicitly defined curves. As we saw in lecture, we can often assume that near a given point, the curve represents the graph of an *implicit* function $y = y(x)$. We can then use implicit differentiation to compute dy/dx , which represents the slope of the tangent line to the curve.

Exercises 3 and 4 in the online version of the lab provide interactive tools to explore this idea. Exercise 3 provides two interactive *SageMath* cells: the first that allows us to find points (x_0, y_0) on the implicit curve by solving an equation, the second—to obtain the implicit derivative dy/dx as a function of x and y . Exercise 4 provides an interactive *SageMath* cell that we can use to plot an implicit curve and up to three other functions $y = y(x)$ on the same plot. While any three graphs can be superimposed on the plot of the implicit curve, Exercise 4 is focused on joint plots of the original curve and its tangent lines at various points.

Exercises

Exercise 5.1. Determine which of the following points lie on the curve

$$\frac{1}{2}y^2 + y - \sin(2\pi x) = \frac{1}{2}x^3y + x.$$

For each part, type your answer and a supporting pair of evaluations in the box below.

- (a) $(x_0, y_0) = (-1, 1)$
- (b) $(x_0, y_0) = (-1, -2)$
- (c) $(x_0, y_0) = (2, 3 + \sqrt{13})$
- (d) $(x_0, y_0) = (2, -3 + \sqrt{13})$

Exercise 5.3. The *SageMath* interactive cell after Exercise 4 in the online version of this lab is initialized to display the curve $y^3 - 16y = x^3 + x$. Use that plot and the two *SageMath* cells in Exercise 3 in the online version of the lab to answer the following questions.

- (a) By adjusting the equation to solve in the *SageMath* equation solver, find the points on the curve where $x = -2$.

- (b) For the points you found in (a), determine the slope of the tangent line to the curve at those points. (The *SageMath* cell below Exercise 3 labeled “Equation for curve” can be used to compute an expression for dy/dx .) In the space below, list the slopes in the following format:

“The slope of the tangent line at (**, **) is ***.”

- (c) While portions of the curve do appear to be close to horizontal, use the expression you found for dy/dx to explain why there are no points on the curve where the tangent line is horizontal.

Notes on Lab 5.

LAB 6

Extreme Values

One of the most important applications of the ideas of calculus is to *optimization*: trying to find the maximum or minimum values of some function that we are interested in. In this lab we will explore how to find these extreme values and some applications.

To find the extreme values of a function, we need to think about what happens to the function at these points. Recall the following definitions from the lectures.

We say a function f has an **absolute maximum** on an interval I at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$; we say a function f has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all $x \in I$. If f has either an absolute maximum or an absolute minimum on I at $x = c$, we say that f has an **absolute extremum** at $x = c$.

We say a function f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x near c (though there may be larger values further away from $x = c$); we say a function f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x near c . Finally, f has a **local extremum** at $x = c$ if it has either a local maximum or a local minimum at that point.

Sometimes it isn't clear from looking at the graph of a function where exactly the largest or smallest values are. In these situations we need to compute the critical points first in order to find the maximum or minimum value of the function.

To do this we are going to need a few tools from *SageMath*. The first is the `diff(f,x)` command for differentiation, which we encountered in Lab #4. Recall that the following *SageMath* code

```
f(x)=x+2*cos(2*x)
df(x)=diff(f,x)
```

defines `df(x)` to be the derivative function of $f(x) = x + 2 \cos(2x)$.

In addition, we will use *SageMath*'s `find_root(f(x),a,b)` command, which returns the value of a root of the function f in the interval $[a,b]$. You can try this by copying the two lines below into the *SageMath* cell in the online version of the lab.

```
f(x)=sin(x)
find_root(f(x),3,4)
```

Note that this command will only return one root in the interval, even if there are multiple roots. Try changing the 3 to a -1 above. The function $f(x) = \sin(x)$ has two roots on the interval $[-1, 4]$, but the `find_root` command only reports one of them.

EXPLORING THE DISTANCE BETWEEN EARTH AND MARS

You may have noticed over the years that Mars occasionally appears especially bright in the night sky. The media certainly tends to make a fuss out of this when it happens. When this does happen, it is because Earth and Mars become unusually close in their orbits around Sun. Because Earth is closer to Sun than Mars, it orbits faster. As a result, the two planets are closest together when Earth passes between Mars and Sun.

The last couple of exercises in this lab lead you through an exploration of a function modeling the distance between Earth and Mars using calculus. Planets orbit approximately in ellipses, but this is only an approximation. The equations below give the approximate (x, y, z) coordinates of Earth and Mars relative to Sun in space over time, measured in days after January 1, 2000; distances are in *astronomic units* (AU)¹. The equations are messy, because orbits are complicated. The important takeaway from these formulas is that $x_{earth}(t)$, $y_{earth}(t)$, $z_{earth}(t)$ are the coordinates of Earth on day t , and $x_{mars}(t)$, $y_{mars}(t)$, $z_{mars}(t)$ are the coordinates of Mars on the same day.

$$\begin{aligned}
 MC &= 0.093405 \\
 MO(t) &= 0.0091458 * t + 0.3246 \\
 MG(t) &= MO(t) + MC * \sin(MO(t)) * (1 + MC * \cos(MO(t))) \\
 MH(t) &= \text{atan2}((0.9956) * \sin(MG(t)), (\cos(MG(t)) - MC)) + 5.0003 \\
 MF(t) &= 1.5236 * \sqrt{(\cos(MG(t)) - MC)^2 + ((0.9956) * \sin(MG(t)))^2} \\
 EC &= 0.016709 \\
 EO(t) &= 6.21419244 + 0.017202 * t \\
 EG(t) &= EO(t) + EC * \sin(EO(t)) * (1 + EC * \cos(EO(t))) \\
 EH(t) &= \text{atan2}(0.99986 * \sin(EG(t)), \cos(EG(t)) - EC) + 4.938241 \\
 EF(t) &= -\sqrt{(1 + EC^2 * \sin(EG(t))^2 + (-2 * \cos(EG(t)) * EC + EC^2))} \\
 x_{earth}(t) &= EF(t) * \cos(EH(t)) \\
 y_{earth}(t) &= EF(t) * \sin(EH(t)) \\
 z_{earth}(t) &= 0 \\
 x_{mars}(t) &= MF(t) * (0.648685933 * \cos(MH(t)) - 0.761056213 * \sin(MH(t)) * 0.999478937) \\
 y_{mars}(t) &= MF(t) * (0.761056213 * \cos(MH(t)) + 0.648685933 * \sin(MH(t)) * 0.999478937) \\
 z_{mars}(t) &= MF(t) * (\sin(MH(t)) * 0.032277747)
 \end{aligned}$$

You will use the above formulas to define a function that approximates the distance, in AU, between Earth and Mars, and use that function to estimate the dates during a given time period when the two planets are at their closest and their furthest. At the end you will compare your findings to the exact distances listed in Table 6.1 at the end of this lab.

¹Recall from astronomy that 1AU, the “distance between Earth and Sun”, is approximately 93 million miles.

Exercises

Exercise 6.1. Exercise 1 in the online version of this lab displays both the graph of a function $f(x)$ on the left and its derivative $f'(x)$ on the right. A slider bar allows you to move a point along the graph of each function. Explore the graph using the slider bar and use it to answer the questions below.

- (a) Find the values of x at which f has an absolute maximum or an absolute minimum over the interval $[0, 4]$. (The horizontal line drawn on the graph of f may be helpful in determining which point it is.)

- (b) Find all the values of x at which f has a local extremum. (Remember the absolute maxima and minima can be local maxima and minima as well.)

- (c) For each local extremum, find the value of the derivative at that point. If the derivative does not exist, write DNE.

Exercise 6.2. Let f be a continuous function defined on a closed bounded interval $[a, b]$. Recall that $x = c$ is called a **critical point** of f if either $f'(c) = 0$ or $f'(c)$ does not exist. For each of the following statements, determine whether it is true or false. Note that to be true, a statement must be true always, but to be false, it suffices to fail just once. If a statement is false, provide a counterexample for that demonstrates that the statement is false.

[**Hint.** Some counterexamples may be found in the function from Exercise 1.]

- (a) If $x = c$ is a local extremum of f , then $x = c$ is a critical point of f .

- (b) If $f'(c) = 0$ then $x = c$ is either a local minimum or a local maximum of f .

- (c) If $f'(c)$ does not exist then $x = c$ is either a local minimum or a local maximum of f .

- (d) The absolute extrema of f on $[a, b]$ are always at one end of the interval, $x = a$ or $x = b$.
- (e) The absolute extrema of f on $[a, b]$ are never at one end of the interval, $x = a$ or $x = b$.
- (f) The absolute maximum of f on $[a, b]$ can always be found by computing $f(x)$ at every critical point as well as the endpoints of the interval and then taking the point where $f(x)$ was largest.
- (g) If $x = c$ is a critical point of f' , then $x = c$ is also a critical point of f .

Exercise 6.3. The function $f(x) = x + 2 \cos(2x)$ has three roots. Use the *SageMath* evaluation cell above Exercise 3 in the online version of this lab to produce a plot of the graph $y = f(x)$ and to identify three intervals that contain those three roots. Then use the `find_root()` command to find all three roots. When finished, save your code in the space below. Note that you will need to run the `find_root()` command three times, changing the interval in which you ask it to search each time. Use the `print()` command to make sure that *SageMath* displays the results of the calculations of all three roots!

Notes on Lab 6.

TABLE 6.1. Closest approaches of Mars to Earth since January 1, 2000.

Date of Closest Approach			Distance (AU)	Days since 01/01/2000
Year	Month	Day		
2001	6	21	0.45017	537.9562
2003	8	27	0.37272	1334.4111
2005	10	30	0.46406	2129.1431
2007	12	18	0.58935	2908.9910
2010	1	27	0.66398	3679.7931
2012	3	5	0.67368	4447.7090
2014	4	14	0.61756	5217.5375
2016	5	30	0.50321	5994.9000
2018	7	31	0.38496	6786.3271
2020	10	6	0.41492	7584.5965
2022	12	1	0.54447	8370.0958
2025	1	12	0.64228	9143.5681
2027	2	20	0.67792	9912.0097
2029	3	29	0.64722	10680.5389
2031	5	12	0.55336	11454.1597
2033	7	5	0.42302	12239.4715
2035	9	11	0.38041	13037.5979
2037	11	11	0.49358	13829.3333
2039	12	28	0.61092	14606.6160

LAB 7

Newton's Method

Newton's method is a way to find the approximate value of a **root** of a function f : the value of x where the function's graph intersects the x -axis. In other words, this method gives an approximation to a solution x^* of the equation $f(x) = 0$. In general, the method starts with an initial guess x_0 for the unknown exact solution x^* and finds a list of increasingly better approximations x_1, x_2, \dots to x^* .

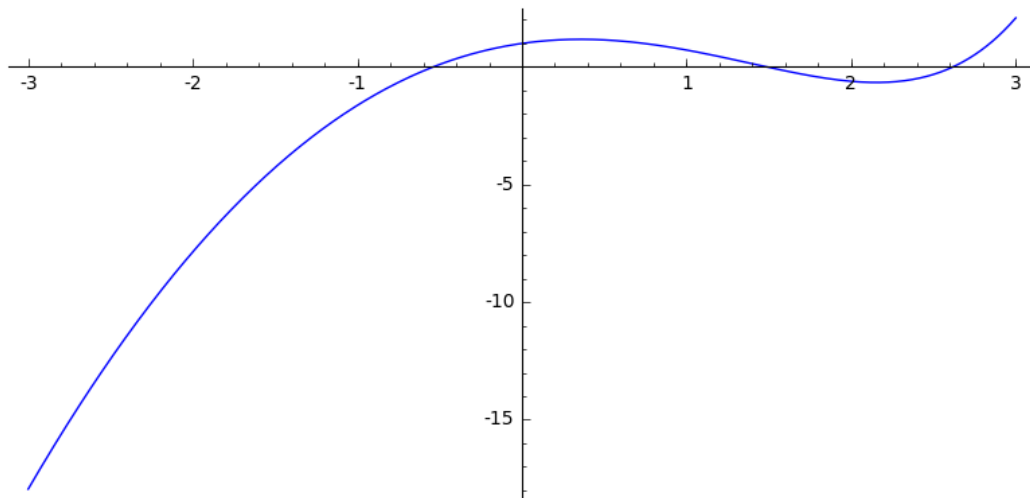


FIGURE 7.1. The graph $y = e^x - 2x^2$.

We will illustrate this method for the function $f(x) = e^x - 2x^2$, which has the graph depicted on Figure 7.1 above. We want to approximate the left-most root, starting at a point somewhat near it. To begin, let us choose $x_0 = -2.2$. This choice and the nearby root can be seen in Figure 7.2, where we have zoomed closer on the graph. The basic idea behind Newton's method is that when x_0 is near x^* the tangent line to the graph $y = f(x)$ at $x = x_0$ will be very close to the actual graph. The x -intercept of the tangent line will be close to the x -intercept of the graph—that is, to the unknown root x^* . Thus, Newton's method determines x_1 as the x intercept of the tangent line at x_0 . The method then recycles the same idea: once we have x_1 , we calculate x_2 as the x -intercept of the tangent line to $y = f(x)$ at $x = x_1$, and so on. The first two steps in this process for our example are shown on the next figure. Each red slanted line segment in the figure is part of the tangent line to $y = f(x)$ at some x_k . We get the next approximation x_{k+1} by taking where the tangent line intersects the x -axis. In this case, as k increases, each term x_k becomes a better approximation to the root x^* . Clearly, we could continue with x_3, x_4, \dots for even better approximations. This idea is summarized by the following algorithm.

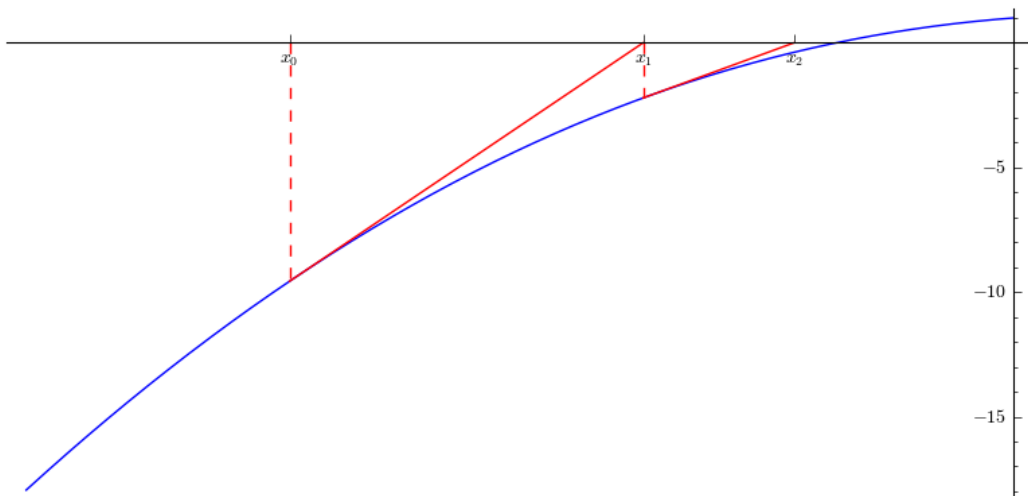


FIGURE 7.2. Two iterations of Newton's method for $e^x - 2x^2 = 0$.

A Step of Newton's Method:

To obtain x_{k+1} from x_k :

1. Find the numerical value of $m = f'(x_k)$, the derivative of f at the point x_k .
2. Using this slope, find the equation of the tangent line $y = mx + b$ to $y = f(x)$ through the point $(x_k, f(x_k))$. (Note that the slope m was determined in Step 1, so now we just need to find the intercept b .)
3. Using these values of m and b , find the value of x where the line $y = mx + b$ intersects the x -axis. This is the value of the next approximation x_{k+1} .

Recall that the equation of the tangent line to $y = f(x)$ at a point $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

When $f'(a) \neq 0$, this straight line intersects the x -axis at

$$x = a - \frac{f(a)}{f'(a)}.$$

When we apply this to $a = x_k$, we conclude that given a k -th Newton iterate x_k , with $f'(x_k) \neq 0$, we can obtain x_{k+1} by the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Now that we understand the method, we may use *SageMath* to speed up calculations. The *SageMath* demo in the online version of the lab allows you to input a function and a starting point x_0 that you want to use Newton's method on. By changing x_{\min} and x_{\max} you can change the part of the graph you see. You can control the number of approximations n computed using the slider bar. You should experiment briefly with it before proceeding with the lab assignment.

Exercises

Exercise 7.1. Let $f(x) = e^x - 2x^2$ and $x_0 = -2.2$. Use the *SageMath* evaluation cell above Exercise 1 in the online version of the lab to perform the following calculations required to determine the values of x_1 and x_2 in Figure 7.2. Save your code in the space below.

- (a) Find the slope of the tangent line to the graph at $x = x_0$.
- (b) Use the slope you found in (a) to find the equation of the tangent line to $y = f(x)$ at $x = x_0$.
- (c) Determine the value of x where the tangent line you found in (b) intersects the x -axis. This is the value of x_1 .
- (d) Starting with x_1 instead of x_0 , repeat the steps in parts (a)-(c) to find the value of x_2 .

Notes on Lab 7.

LAB 8

Riemann Sums

APPROXIMATING AREAS WITH RIEMANN SUMS

In this lab, we will explore how to approximate the area of a bounded region R , like the shaded blue region in Figure 8.1 below. One way to do this is to use rectangles to approximate the area of R . When we use the area of rectangles to approximate the area of a bounded region we compute what is called a *Riemann sum*. To compute a Riemann sum, we follow the following algorithm:

1. Partition the interval $[a, b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$.
2. Generate n rectangles of equal widths Δx .
3. The height of each rectangle is determined by a value of the function over that subinterval. We can use the value of the function at the left endpoint of the subinterval, at the midpoint of the subinterval, at the right endpoint of the subinterval, or at a random x value in that subinterval. We call these values of x *sample points*.
4. Put together, the rectangles form a polygon whose area approximates the area of R .

For example, we can approximate the area of the region bounded by $f(x) = 3x^2 + 2$, the horizontal axis, and the lines $x = 0$ and $x = 2$ with 4 rectangles. Then each rectangle would have a

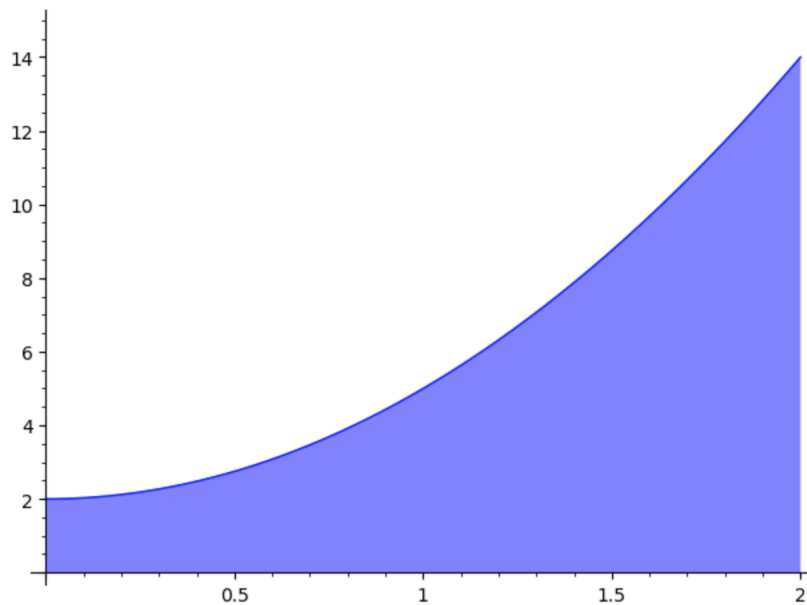


FIGURE 8.1. The region below the curve $y = 3x^2 + 2$.

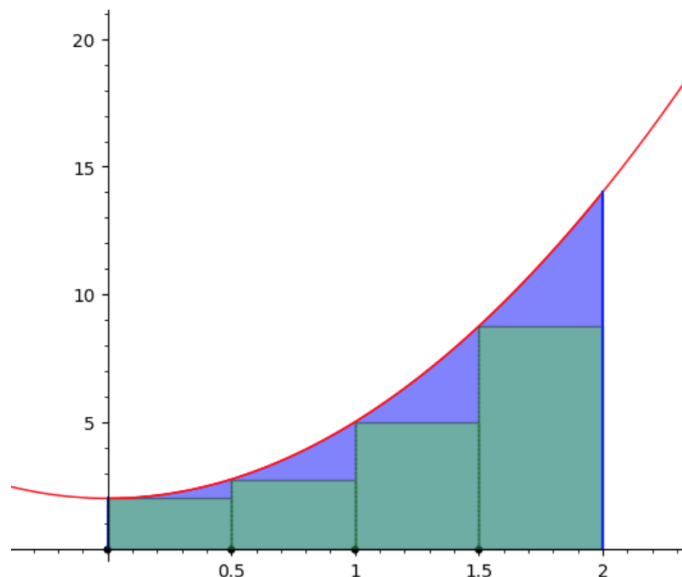


FIGURE 8.2. A left Riemann sum with four rectangles approximating the area below the curve $y = 3x^2 + 2$.

width of $\Delta x = \frac{2-0}{4} = 0.5$. Such an approximation based on the values at the left endpoints is shown on Figure 8.2. We can use *SageMath* to help us compute a left Riemann sum by adding together the area of each rectangle.

$$f(x) = 3x^2 + 2$$

$$f(0) \cdot (0.5) + f(0.5) \cdot (0.5) + f(1) \cdot (0.5) + f(1.5) \cdot (0.5)$$

LIMITS OF RIEMANN SUMS

To get better approximations to different areas, you will experiment with different choices for the number and locations of the sample points. In your experiments, you will use an interactive *SageMath* demonstration in the online version of the lab that allows you to select: the function f whose graph bounds the region R from above, the interval $[a, b]$ that bounds R from below, the number n of subintervals that we subdivide $[a, b]$ into, and a rule for the sample points. There are four possible rules you can choose from:

- the sample point is always the left endpoint of the subinterval: these lead to “left Riemann sums”;
- the sample point is always the right endpoint of the subinterval: these lead to “right Riemann sums”;
- the sample point is always the midpoint of the subinterval;
- the sample point in each subinterval is chosen at random within that subinterval.

The demonstration provides also a button to regenerate the chosen Riemann sum. In the case of a Riemann sum with randomly chosen sample points, this button reselects the sample points and generates a different randomized Riemann sum based on the same partition of $[a, b]$.

Exercises

Exercise 8.1. In the space below, write *SageMath* commands to evaluate the Riemann sum for $f(x) = 3x^2 + 2$ on $[0, 2]$ with 5 rectangles using right-hand endpoints. Record your approximation in your lab notebook. State explicitly the value of Δx in this example.

Exercise 8.5. Let $f(x) = 3x^2 + 2$ and answer the following questions.

(a) Find the antiderivative g of f : that is, find a function g such that $g'(x) = f(x)$ for all x in the given interval $[a, b]$. In the space below, write *SageMath* code that checks that your choice for g has the desired property. (Recall that *SageMath*'s command `diff(func, x)` computes the derivative $\text{func}'(x)$.)

(b) Use the *SageMath* evaluation cell in the online version of the lab to evaluate $g(a)$, $g(b)$ and $g(b) - g(a)$. Record those values below. (Use *SageMath*'s command `print()` to display the results of multiple computations.)

Exercise 8.6. Based on all the evidence collected throughout the lab: How does the data recorded in part (b) of Exercise 5 relate to the values of the limits recorder in part (b) of Exercise 3 of the online version of the lab?

Notes on Lab 8.