

## A SEMIREDUCTIBLE APPROACH TO THE 5D GRAVITY.

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**Introduction.** The possibility that matter has a purely geometric origin [13]<sup>1)</sup> is an attractive idea, which has found one of its best formulations in the five dimensional theories of (variable) gravity (shortly 5D gravity). Originally the fifth dimension was associated with the variable mass of the particle [29], [30], but the theory itself can be worked out quite independently of this hypothesis. In particular, the fifth dimension is *not* required to be compact and hence unobservable and it can be regarded as a convenient device which induces “matter” in the four dimensional Universe of our perception. In fact, the extra terms in the 5D Einstein tensor, which depend on the fifth scalar factor or on the derivatives of the fifth coordinate can be suitably identified in many cases, with the “matter density” and “pressure”. The aforementioned procedure prescribes a perfect fluid and the theory is compatible with both the Newton’s law of motion and the first law of thermodynamics (see [33], [34], [35] for details). Furthermore it has been recently argued that theories of this type are equivalent or almost equivalent to brane-world theories [25]. Therefore, a re-examination of such theories from a rather different point of view is in order. In particular, it is interesting to notice that although there are many known solutions of the 5D gravity, an overall classification scheme of these solutions, which permits an immediate more or less comparison with 4D physics, is missing.

The main objective of this paper is a reconsideration of the mathematical background of the 5D gravity and the introduction of an effective and convenient classification scheme. The possibility of such a scheme is hinted in a recent work on the Einstein spaces of any dimension  $n \geq 4$ , which are of the semireducible type [10]. In this context we recall that semireducible spaces form a rather particular class of Riemannian spaces (see for details [19], [20]). Here, the way semireducibility is involved in 5D gravity (one possibility is through those solutions that happen to be  $V(0)$ -spaces) is investigated and the situation is further clarified. We stress that our approach is different from that of Romero et al who use the Campbell-Magaard theorem [27]. Here, emphasis is given into writing down the 5D metric into an appropriate *semireducible* form in a way such that the field equations of the 5D gravity are considerably simplified. This approach enables us to treat effectively the non-vanishing off-diagonal terms, which are now hidden into an appropriate diagonal block, and in addition to identify those solutions, which are of the  $V(0)$ -type. In the first case a comparison of our approach to that of the “splitting technique” of Ponce de Leon [26], is of some interest. In our case, we are led into a simplified set of field equations while in his

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Received February 28, 2004 (EH paper).

1) Numbers in brackets refer to the references at the end of the paper.

case “the most general induced energy-momentum tensor” is obtained. However, our second goal, namely the identification of the solutions of the  $V(0)$ -type for the various semireducible types, illuminates an essentially *new* phenomenon, which is characteristic of any multidimensional theory of gravity ( $n > 4$ ). We call this phenomenon the inertial degeneracy of the  $n$ D vacuum and we demonstrate explicitly its existence (in the case of the 5D gravity) in §3. Last but not least an overall classification scheme is presented in §4.

**§ 1. Semireducible Riemannian spaces.** A Riemannian space will be called semireducible, if to a suitably chosen coordinate system, its line-element  $ds^2$  can be written in the form

$$(1.1) \quad ds^2 = ds_0^2 + \sigma_1^2 ds_1^2 + \dots + \sigma_q^2 ds_q^2,$$

where  $\sigma_\alpha \neq c\sigma_\beta$  ( $\alpha \neq \beta$ ,  $c = \text{const.}$ ) and

$$ds_a^2 = g_{i_a j_a} dx^{i_a} dx^{j_a}, \quad (a = 0, 1, \dots, q).$$

Here, each one of the appended metrics  $ds_\alpha^2$  ( $\alpha = 0, 1, \dots, q$ ) depends only on the corresponding internal coordinates  $\{x_\alpha\}$ , while the functions  $\sigma_\alpha$  depend only on the coordinates  $\{x_0\}$  of the fundamental part, or kernel,  $ds_0^2$ , of the above line element. If one at least of the  $\sigma_\alpha$  is constant, the space is said to be reducible or decomposable. The above analysis is called the  $q$ -analysis of the semireducible space  $V_n$  ( $n > 2$ ). The  $q$ -analysis considered above is not unique, in general (see [19], [20]). However, assuming that 1-dimensional appended metrics are permissible, we may conclude that a unique  $q_{max}$  is always possible ( $1 \leq q_{max} \leq n - 1$ ), but again the corresponding  $q_{max}$ -analysis need not be unique. Nevertheless  $q_{max}$  is a characteristic invariant of the semireducible space under consideration. In all cases, the particular  $q$ -analysis in use, implies a certain decomposition of the dimension, according to the formula

$$(1.2) \quad n = p + \sum_{a=1}^q n_a,$$

where  $p$  is the dimension of the kernel and  $q$  is the *length* of the analysis under consideration. The  $n_\alpha$  ( $\alpha = 1, \dots, q$ ), are the dimensionalities of the appended metrics or *slices* of our analysis. Clearly,  $1 \leq n_\alpha \leq n - p$ .

Equation (1.2) provides a convenient classification scheme for any  $q$ -analysis. In particular, the arbitrary  $q$ -analysis of the semireducible space  $V_n$ , is completely prescribed by the integer  $p$  ( $p < n$ ) and by the partition of the integer  $n - p$ . The length of the partition of  $n - p$  is obviously the integer  $q$ . The *rank* of the partition<sup>2)</sup> is also important for the characterization of the  $q$ -analysis under consideration. In the following we shall not use the Frobenius nomenclature for partitions, being too complicated although very powerful. Instead, we shall use the sequence

$$(1.3) \quad \{p, k_1, 2(k_2), \dots, k_p\}.$$

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2) See Littlewood [22], p. 60.

where the first integer is, in all cases, the dimension of the kernel and the rest represents, the dimensions of the slices in descending order. Repetitions are explicitly illustrated, using brackets. For example in the case of (1.3) we have two distinct slices of the same dimension  $k_2$ . Obviously, we may redefine the kernel by absorbing a certain number of slices in it. The end product of this procedure is that some of the internal coordinates change status becoming coordinates of the kernel and they can be regarded as dummy, since they do *not* appear in the  $\sigma_\alpha$  of the new analysis, which is necessarily a  $(q - k)$ -analysis ( $k < q$ ). Hence, the resulting new analysis is *reducible*<sup>3)</sup> or degenerate. Nevertheless, this possibility is important as far as the multidimensional theories of gravity are concerned, since it indicates that we may use two “different” sets of field equations in order to get the same solution.

The interesting point, for our considerations, is that in all cases the field equations are considerably simplified to the form

$$(1.4) \quad (a) \quad R_{ij}^{(0)} = \sum_{a=1}^q n_a \sigma_a^{-1} \sigma_{a;ij}, \quad (b) \quad R_{ij}^{(a)} = \frac{R_a}{n_a} g_{ij}^{(a)},$$

where the constants  $R_a$  are given from the equations

$$(1.4) \quad (c) \quad \frac{R_a}{n_a} = (n_a - 1) \Delta_1 \sigma_a + \sigma_a \Delta_2 \sigma_a + \sigma_a \sum_{b \neq a} n_b \sigma_b^{-1} \Delta_1(\sigma_a, \sigma_b)$$

and

$$\Delta_2 \sigma = g^{ij} \sigma_{;ij}.$$

Here,  $R_{ij}^{(0)}$  and  $R_{ij}^{(\alpha)}$  represent the Ricci tensor of the kernel and of the various slices respectively. The covariant derivatives refer exclusively to the kernel. In particular, equation (1.4)(b) implies that the appended metrics are necessarily Einstein spaces of scalar curvature  $R_\alpha$ , where the constants  $R_\alpha$  are prescribed from the solution of the combined set of the equations.

The above equations are a special case of the equations given in [10] and they can be immediately obtained from those, for  $R = 0$ . The tremendous simplification here is that instead of the  $n$ -dimensional vacuum field equations of the  $n$ D gravity we have only to solve equations (1.4)(a) subject to the conditions (1.4)(c), which live in the  $p$ -dimensional kernel of our semireducible analysis (1.1). Since  $p < n$ , in all cases, the induced simplification is obvious. Further to this point the off-diagonal terms, which are the headache of the multidimensional theories, can be most conveniently hidden either in the kernel or in one of the slices of the adopted  $q$ -analysis.

**§ 2. The  $V(K)$ -spaces.** A special class of semireducible spaces is that of the  $V(K)$ -spaces. The Einstein spaces of Brinkmann are a special instance of these spaces of the type  $\{1, n - 1\}$  ([2], see also [23], p. 81). Some of the  $V(K)$ -spaces have been also presented by De Vries, as solutions to the problem of finding the spaces that

3) Here the term “reducible” refers to the  $q$ -analysis under consideration and not to the underlying space. The term degenerate means that the very same  $q$ -analysis can be written using different  $p$  and  $q$  in each case.



admit a gradient conformal transformation [12]. Although the results of De Vries generalize considerably the Brinkmann's solution, the geometric properties and the identification of the  $V(K)$ -spaces as a special class of pseudo-Riemannian spaces is due to the systematic investigations of Solodovnikov in 1956 (see [28], [21] and references therein). For our purposes these spaces are very interesting because they provide almost automatically a broad class of possible solutions of the 5D gravity.

In a  $V(K)$ -space, the kernel ( $p > 1$ ) is always a *space of constant curvature*, but in addition the functions  $\sigma_\alpha$  are not arbitrary any more. In fact they are determined from the conditions [28],

$$(2.1) \quad \begin{aligned} (a) \quad \sigma_{a;ij} &= -K\sigma_a g_{ij}^{(0)}, \\ (b) \quad \Delta_1(\sigma_a, \sigma_b) &= -K\sigma_a \sigma_b, \quad (a \neq b), \end{aligned}$$

where  $K$  is a real constant, characteristic of the space under consideration. Here,  $\Delta_1$  is the usual differential parameter of the first order, namely

$$\Delta_1(\sigma_a, \sigma_b) \equiv g^{ij} \partial_i \sigma_a \partial_j \sigma_b.$$

We stress that equations (2.1)(a) and (2.1)(b) hold for  $p > 1$ . For  $p = 1$  equation (2.1)(a) is replaced by the equation

$$(2.2) \quad \ddot{\sigma}_a + \varepsilon K \sigma_a = 0, \quad (\varepsilon = \pm 1),$$

where  $\varepsilon$  indicates the sign of the 1-dimensional kernel in (1.1). In writing (2.2) we have normalized the scaling factor of the 1-dimensional kernel. A more general expression can be also given. The  $q$ -analysis of a  $V(K)$ -space is usually called a  $K$ -analysis or  $K$ -factorisation of the  $V(K)$ -space under consideration. Clearly, in a  $V(K)$ -space the slices are quite arbitrary. Thus, a  $V(K)$ -space is the prototype of semireducible spaces which define a  $q$ -structure. In our case this  $q$ -structure is completely defined by the above equations and for any  $p \geq 1$ .

For our purposes the crucial point is that the quantities

$$(2.3) \quad K_a = \Delta_1 \sigma_a + K \sigma_a^2,$$

are constants assuming only that equation (1.1) is a  $K$ -analysis. Following Solodovnikov we shall call the constant  $K_a$  the *conjugate curvature* of the slice  $\alpha$ . We shall not get deeper in the Mathematics of the  $V(K)$ -spaces. The only important theorems we shall need in the sequel are:

**Theorem 1.** *The adjoint metric of the  $q$ -analysis (1.1) (i.e. the metric which results from (1.1) by considering 1-dimensional slices) in a  $V(K)$ -space is always that of a space of constant curvature  $K$ .*

*Proof.* This is an immediate consequence of an alternative definition of a  $V(K)$ -space, used by Solodovnikov in [28]. However, a direct calculation of the curvature tensor of the adjoint metric that is derived from the metric (1.1), using equations (2.1)(a) and (2.1)(b) proves directly our assertion.

**Theorem 2.** *The semireducible metric (1.1) represents a space of constant curvature  $K$ , iff*

1. *The  $q$ -analysis (1.1) is a  $K$ -analysis.*
2. *Each appended metric-form  $ds_a^2$  has a constant curvature  $K_a$*

(see [28] and [21] for the proof).

**Theorem 3.** *The semireducible metric (1.1), of a  $V(K)$ -space, represents a non-trivial Einstein space of scalar curvature  $n(n-1)K$ , iff each appended metric-form  $ds_a^2$  is that of an Einstein space of the scalar curvature  $n_a(n_a-1)K_a$ , where  $K_a$  is the conjugate curvature of this form and at least one of the aforementioned spaces is non-trivial (i.e. it does not represent a space of constant curvature (see [28], [9] for details and the proof).*

**Theorem 4.** *Any solution of the 5D gravity of the partition type  $\{1, 4\}$ , is necessarily a  $V(0)$ -space.*

*Proof.* From the field equations (1.4)(a) and for  $p = q = 1$ , we conclude that the function  $\sigma$  satisfies equation (2.2) for  $K = 0$ . In addition  $q = 1$ . Hence, equation (2.1)(b) is identically satisfied and the solution under consideration is, according to our definition, a  $V(0)$ -space. Furthermore, we can always normalize the function  $\sigma$  so that the conjugate curvature is  $\varepsilon$ . This completes the proof.

We may now use the above theorems and in particular Theorems 3 and 4 to construct formally all the non-trivial solutions, which are of the type  $\{1, 4\}$ . In fact, the requirement that the solution is of the aforementioned type implies ( $K = 0$ ) that the solution is of the general form

$$(2.4) \quad ds^2 = \varepsilon du^2 + u^2 ds_1^2,$$

where  $\varepsilon = \pm 1$ . In addition, according to Theorem 3 of this section, the appended metric in (2.4) is necessarily a four dimensional Einstein space, the scalar curvature of which is  $n_1(n_1-1)K_1$ , where  $K_1 = \varepsilon$  and  $n_1 = 4$ . This information is sufficient for our construction, i.e. it is sufficient to choose any 4-dimensional Einstein space of the aforementioned curvature and then *replace*  $ds_1^2$  in (2.4) by the metric of this particular space, in order to have a solution of the 5D gravity, which is by construction a  $V(0)$ -space. In the particular case, where the slice  $ds_1^2$  is replaced by the Kottler solution the Henriksen-Emslie-Wesson metric is obtained (see [34], [18]). However, in principle, there are infinitely many candidates for the aforementioned slice, which result into different solutions. An interesting example can be obtained by replacing the slice in (2.4) by the solutions prescribed by Petrov in 1946 (see [23] p. 84 and [10]). In all cases the induced 4D energy-momentum tensor prescribes the “signature” of the 4D vacuum. In particular, for any 4D hypersurface  $u = u_0 = \text{const} \neq 0$ , of the bulk space (2.4), the induced energy-momentum tensor, up to a choice of units and constants, is of the form

$$T_{ij}^{(4)} = -3\varepsilon u_0^{-2} g_{ij}^{(4)},$$

where  $\varepsilon = \pm 1$  prescribes the sign of the extra dimension.

**§ 3. Inertial degeneracy.** The main objective of this section is to reconsider the “signature” of the 5D vacuum from an essentially different point of view. In this context we recall that originally, in 1984 [30], the field equations of the 5D theory incorporated a 5-dimensional energy momentum tensor  $T_{ij}$ . Later, in 1988, it occurred independently to Wesson, Ponce de Leon and Coley that the appropriate field equations are  $G_{ij} = R_{ij} = 0$  rather than  $G_{ij} = T_{ij}$ , the logical conclusion being that the 5D theory may not really need an explicit energy momentum tensor (see for details [32]). In 2002 the 5D energy-momentum tensor reappears in a rather formal approach by Ponce de Leon [26]. We shall call these two, essentially different, approaches to the 5-dimensional gravity, as the W1 and the W2 versions respectively. In the W2 version the 5D energy-momentum tensor identically vanishes, while in the W1 version this is not the case. We shall explicitly demonstrate in the sequel that these two versions of the 5D gravity are related by a geodesic correspondence of the underlying spaces, which is non-trivial at least in all cases, where solutions of the  $\{1, 4\}$  partition type are involved. The implications of this observation are rather important. Roughly speaking geodesics, with the exception of few trivial cases, determine uniquely the geometry of space-time of General Relativity ([23], chapter 7), whereas our demonstration proves that geodesics do *not* determine uniquely the geometry of the 5D vacuum. There are more than one dynamical setting (in our case the W1 and the W2 versions of the 5D gravity of the partition type  $\{1, 4\}$ ), which both prescribe the same geodesics, i.e. the same orbits of freely falling test-particles and light rays. We call this phenomenon the *inertial degeneracy* of the 5D vacuum.

We complete our discussion giving a constructive proof for the existence of non-trivial solutions of the 5D gravity, which manifest the aforementioned phenomenon. In particular, we may consider the special 1-parameter family of  $(n + 1)$ -dimensional spaces ( $n \geq 4$ )

$$(3.1) \quad ds^2(\lambda) = \frac{\varepsilon du^2}{(1 + \lambda u^2)^2} + \frac{u^2}{1 + \lambda u^2} ds_1^2.$$

Here the real parameter  $\lambda$  is quite arbitrary and  $u$  represents the additional fifth dimension. Using Theorem 1 of §3 we can easily prove that the family (3.1) is a family of  $V(K)$ -spaces with the constant  $K$ , of each member of the family, given by the expression

$$(3.2) \quad K(\lambda) = \varepsilon \lambda.$$

Further to this point for  $\lambda = 0$  we have a  $V(0)$ -space of the form (2.4), which is a solution of the W2 version of the 5D gravity. Then we can easily prove, after some straightforward algebra, that for each  $\lambda \neq 0$  the corresponding space of the family (3.1) is in (local) geodesic correspondence with our aforementioned solution. Thus we have proved directly that there is an infinity of spaces, which are in a geodesic correspondence to a  $V(0)$ -solution of the W2 type of the 5D gravity. These spaces can be regarded as solutions of the W1 version of the 5D gravity since their corresponding energy-momentum tensor is uniquely defined by the aforementioned procedure. In addition from equation (2.3) we can easily calculate the conjugate curvature

$$K_1(\lambda) = \varepsilon,$$



which is the same for all the members of the family of spaces (3.1). This is an interesting result since it implies, as a consequence of Theorem 3, that if the  $\lambda = 0$  member of the family (3.1) is a (special) Einstein space then all the members of this family are Einstein spaces and their scalar curvature is  $R = n(n-1)K(\lambda)$ , where  $K(\lambda)$  is given by (3.2). We stress that in this case, as well as in the case of equation (3.1), our solution is not just a  $V(0)$ -space with  $p = 1$ , but a class of infinitely many *distinct* spaces.

**§ 4. Classification of the solutions of the 5D gravity.** So far we have considered only solutions of the partition type  $\{1, 4\}$ . However our previous analysis and in particular Theorem 4 enable us to know exactly the partition type of solutions, where non-trivial solutions of the  $V(0)$ -type may exist. In particular,  $V(0)$ -solutions of the types  $\{2, 3\}$  and  $\{3, 2\}$  are necessarily spaces of constant curvature as a result of Theorem 3. In addition solutions of the partition types, which prescribe 1-dimensional slices cannot be non-trivial  $V(0)$ -spaces as a result of either Theorem 3 or of Theorem 1. In the case  $\{4, 1\}$  this can be also proved by a direct calculation, i.e. any  $V(0)$ -space of this particular type is flat<sup>4)</sup>. Another way to see this is by the observation that in this case the adjoint metric and the original semireducible metric coincide. Thus, we have proved the following

**Theorem 5.** *Any non-trivial  $V(0)$ -solution of the 5D gravity is necessarily of the partition type  $\{1, 4\}$ .*

We may now classify, by inspection, all the known solutions of the 5D gravity, which belong to anyone of the 11 genuine semireducible types. The number 11 of these classes results in the following way. For  $n = 5$ , we let  $p$ , i.e. the dimension of the kernel to vary from the value 1 to the value 4 and in each case we count the number of possible partitions of the integer  $5 - p$ . In fact, for  $p = 1$  we have five types of solutions, for  $p = 2$  three types, for  $p = 3$  two types and for  $p = 4$  just one type of solutions. Thus, there is a total of *eleven* distinct classes of solutions, which can be written (in a suitably chosen coordinate system) in the semireducible form (1.1). For example, in the case of the partition type  $\{1, 4\}$  belong the static solutions with a 3-dimensional spherically symmetric section. The generalized de Sitter solution can be also regarded as a solution of this type. The most remarkable property of this case is that if a space of this type satisfies the vacuum field equation then it is necessarily a  $V(0)$ -space (Theorem 4 of §2). The solution given by Wesson [31] in 1986, the solution given by Gron [15] in 1988 and the solution presented by Belinsky and Verdaguer [1], belong to the type  $\{1, 3, 1\}$ . Extended Kasner's solutions can be found, which belong to the partition types  $\{1, 2, 2\}$  and to the type  $\{1, 1, 1, 1, 1\}$  (compare with the solutions presented in [10], [16] and [8]). An interesting solution of Davidson and Owen in 1985 and its locally equivalent solution of Chatterjee in 1990 belong to the type  $\{1, 2, 1, 1\}$  (see the references [11] and [5] respectively). The solution of Fukui in 1987 as well as the solutions considered by Ponce de Leon in 1988 and by Chi in 1990 and the solution of Chatterjee in 1987 belong to the type  $\{2, 3\}$  (see the references [14], [24], [7] and [4] respectively). The solutions of Billyard and

4) This holds for any  $V(0)$ -space of the type  $\{n-1, 1\}$ .

Wesson in 1996 as well as the solutions of Wesson, Liu and Lim in 1993 belong to the type partition type  $\{2, 2, 1\}$  (see [3] and [36] respectively). Clearly, these solutions are not degenerate and they should not be confused with those of the type  $\{1, 2, 1, 1\}$ . Last but not least the solution by Chatterjee, Wesson and Billyard in 1997 as well as the solution of Halpern in 2000 belongs to the type  $\{2, 1, 1, 1\}$  (see the references [6] and [16], [17] respectively).

Obviously, the aforementioned classification scheme applies equally well to the general relativity. In this case there are only six distinct types<sup>5)</sup> of semireducible solutions. However, using Theorem 3 and the well-known fact that the 3-dimensional Einstein Spaces are necessarily spaces of constant curvature we conclude that the partition type  $\{1, 3\}$  is trivial in general relativity in the absence of matter ( $T_{ij} = 0$ ), while the corresponding  $\{1, 4\}$  type of the 5D gravity includes the non trivial  $V(0)$ -solutions we have mentioned previously and exhibits the phenomenon of inertial degeneracy as we have proved in §3. An open question remains, namely, if the phenomenon of inertial degeneracy manifests itself in cases where the partition type is different from the type  $\{1, 4\}$ . We conjecture, using a theorem presented by Solodovnikov in [28], that inertial degeneracy may also occurs for the partition types  $\{2, 3\}$  and  $\{3, 2\}$  too. Although a rigorous proof or a concrete example for this statement is required, our previous analysis suggests that 5D gravity is essentially different from the 4-dimensional independently of the version, which is adopted.

*Acknowledgements.* We wish to thank professor C. N. Ktorides for his valuable remarks on the manuscript and professor G. Contopoulos for his comments on the final form of this paper. The first of the authors wishes also to thank professor N. Antoniou and professor P. Varotsos for their encouragement on this time-consuming task.

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## REFERENCES

- [1] V. Belinski and E. Verdaguer : Gravitational solitons, *Cambridge University Press*, 2001, p. 161.
- [2] H. W. Brinkman : Einstein spaces which are mapped conformally on each other, *Math. Ann.*, **94**(1925), 119-145.
- [3] A. Billyard and P. S. Wesson : Class of exact solutions in 5D gravity and its physical properties, *Phys. Rev. D*, **53**(1996), 731-737.

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<sup>5)</sup> Namely  $\{1, 3\}$ ,  $\{1, 2, 1\}$ ,  $\{1, 1, 1, 1\}$ ,  $\{2, 2\}$ ,  $\{2, 1, 1\}$ ,  $\{3, 1\}$ .



- [4] S. Chatterjee : De Sitter-type of cosmological model in a five dimensional theory of gravity with variable rest mass, *Astron. Astrophys.*, **179**(1987), 1-2.
- [5] S. Chatterjee : Static spherically symmetric solution in a Kaluza-Klein type of metric, *Astron. Astrophys.*, **230**(1990), 1-2.
- [6] S. Chatterjee, P. J. Wesson. and A. Billyard : Cylindrically symmetric space-times in 5D and their effective properties of matter, *Il Nuovo Cimento*, **112B**(1997), 1667-1673.
- [7] L. K. Chi : New cosmological models in a five-dimensional space-time-mass gravity theory, *G. R. G.*, **22**(1990), 1347-1350.
- [8] A. Chodos and S. Detweiler : Where has the fifth dimension gone?, *Phys. Rev. D*, **21**(1980), 2167-2170.
- [9] J. P. Constantopoulos : A particular class of Einstein spaces, *J. Phys. A*, **26**(1993), L773-L776.
- [10] J. P. Constantopoulos : Semireducible Einstein spaces, *Proc. of Academy of Athens*, **75**(2000), 113-128.
- [11] A. Davidson and D. A. Owen : Black Holes as windows to extra dimensions, *Phys. Letters B.*, **155**(1985), 247-250.
- [12] H. L. De Vries : Uber Riemannsche Raume, die infinitesimale Konforme Transformationen gestatten, *Math. Zeitschr. Bd. S.*, **60**(1954), 328-347.
- [13] A. Einstein : The meaning of relativity, *Princeton Univ. Press*, 1956.
- [14] T. Fukui : A cosmological solution of a 5D space-time-mass gravity in vacuum, *G. R. G.*, **19**(1987), 43-46.
- [15] O. Gron : Inflationary cosmology according to Wesson's gravitational theory, *Astr. Astrophys.*, **193**(1988), 1-4.
- [16] P. Halpern : Behavior of Kasner cosmologies with induced matter, *Phys. Rev. D*, **63**(2000), 1-6.
- [17] P. Halpern : Exact solutions of five dimensional anisotropic cosmologies, *Phys. Rev. D*, **66**(2002), 1-11.
- [18] R. N. Henriksen, A. G. Emslie and P. S. Wesson : Space-times with constant vacuum energy density and a conformal Killing vector, *Phys Rev. D*, **27**(1983), 1219-1227.
- [19] G. L. Kruckovic : On motions in semireducible spaces, *Usp. Mat. Nauk. (N. S.)*, **12**(1957), 149-156.
- [20] G. L. Kruckovic : On a class of Riemannian spaces, *Trudy Sem. Vector. Tensor Anal.*, **11**(1961), 103-128.
- [21] G. L. Kruckovic : The spaces  $V(K)$  and their geodesic transformations, *Trudy Uzesoju, Zaocn. Energ. Inst.*, **33**(1967), 3-18.
- [22] E. D. Littlewood : The Theory of group characters, 2<sup>nd</sup> edition, *Oxford Press*, 1958.
- [23] A. Z. Petrov : Einstein spaces, *Pergamon Press*, 1969.
- [24] J. Ponce de Leon : Cosmological models in a Kaluza-Klein theory with variable rest mass, *G. R. G.*, **20**(1988), 539-550.

- [25] J. Ponce de Leon : Equivalence between space-time-matter and brane-world theories, *Mod. Phys. Lett. A*, **16**(2001), 2291-2304.
- [26] J. Ponce de Leon : The effective energy-momentum tensor in Kaluza-Klein gravity with large extra dimensions and off-diagonal metrics, *Int. J. Mod. Phys. D*, **11**(2002), 1355-1380.
- [27] C. Romero, R. Tavakol and P. Zalaletdinov : The embedding of general relativity in five dimensions, *G. R. G.*, **28**(1996), 365-376.
- [28] A. S. Solodovnikov : Projective transformations of Riemannian spaces, *Usp. Mat. Nauk (N. S.)*, **11**(1956), 45-116.
- [29] P. S. Wesson : A new approach to scale invariant gravity, *Astron. Astrophys.*, **119**(1983), 145-152.
- [30] P. S. Wesson : An embedding for general relativity with variable rest mass, *G. R. G.*, **16**(1984), 193-203.
- [31] P. S. Wesson : Astrophysical data and cosmological solutions of a Kaluza-Klein theory of gravity, *Astron. Astrophys.*, **166**(1986), 1-3.
- [32] P. S. Wesson : Clarification of an extended theory of gravity and a reply to Gron and Soleng, *G. R. G.*, **22**(1990), 707-712.
- [33] P. S. Wesson : A physical interpretation of a Kaluza-klein cosmology, *Astrophys. J.*, **394**(1992), 19-24.
- [34] P. S. Wesson : Space-Time-Matter, *World Scientific*, 1999.
- [35] P. S. Wesson and J. Ponce de Leon : Kaluza-Klein equations, Einstein's equations and an effective energy-momentum tensor, *J. Math. Phys.*, **33**(1992), 3883-3887.
- [36] P. S. Wesson., H. Liu and P. Lim : Time-dependent Kaluza-klein solitons and their effective properties of matter, *Phys. Letters B*, **298**(1993), 69-72.