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ON THE FLATNESS OF CERTAIN SOLUTIONS OF THE 5D GRAVITY.

By John P. CONSTANTOPOULOS and Antony A. KRITIKOS.

Introduction. The main objective of this note is a systematic and complete (from the Mathematical point of view) reexamination of the "Exact solution",

$$dS^2 = \frac{l^2}{L^2}dt^2 - [l\sinh(\frac{t}{L})]^2 d\sigma_3^2 - dl^2, \tag{1a}$$

$$d\sigma_3^2 = (1+\frac{kr^2}{4})^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \quad \text{and} \quad k=-1, \tag{1b}$$

presented recently by Tomas Liko and Paul S. Wesson [**6**][1)], in the coordinates t, r, θ, ϕ and l. Solutions of this type have some interesting applications according to the aforementioned authors. Thus, revealing its mathematical features might be of particular interest. More explicitly the solutions of this kind belong to the type $\{1,4\}$ according to a certain classification scheme, which was presented in reference [**3**] and they are either flat or $V(0)$-spaces, as has been explicitly demonstrated, among others, in this reference. It is also worth noticing that Liko and Wesson in presenting their solution state that: (1) "satisfies the Ricci-flat equation $R_{AB}=0$ may be shown by tedious algebra... and confirmed by computer". Furthermore, the same authors in the same reference declare that "the only practical way to show that (1) also satisfies the Riemann flat equation $R_{ABCD}=0$ is by computer as may be verified". However, both the above statements can be easily and *analytically* proved, by two theorems presented in reference [**3**] and for the broader class of solutions classified as solutions of the type $\{1,4\}$. We also think that the classification scheme of the 5D gravity presented there might be of further use in the future. The reason we can get such results without much labor is due to the fact that solutions of the $\{1,4\}$ type are particular cases of the $V(K)$-spaces the origin of which goes back to a work of De Vries in 1954 [**8**]. The various geometric features of the aforementioned spaces have also been recognized by the systematic investigations of Solodovnikov and Kruckovic and the details can be found in references [**5**] and [**7**]. The main contribution of reference [**3**] is that it reorganizes the full mathematical background, due to De Vries, Solodovnikov and Kruckovic, into an integrated classification scheme suitable for the 5D gravity as well as for any theory of Gravity in dimension $N>5$ that follows the conceptual scheme of Wesson et al.

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1) Numbers in brackets refer to the references at the end of the paper.

In the light of the above and assuming that there is a specific physical interest in solution (1), we shall reconsider the analytic proof of both statements, i.e. the statement that (1) is Ricci flat and the statement that (1) is a (locally) flat space, in the context of the theory presented in [**2**] and [**3**]. We shall also indicate how the theory applies in more complicated situations, where solutions of the 5D gravity of the type $\{1,4\}$ are encountered.

§ 1. **Preliminaries.** We recall that a $V(K)$-space is a semireducible space the adjoint metric of which is that of a space of the constant curvature K (see [**1**], [**2**] and [**3**]). A $V(K)$-space degenerates into a space of constant curvature if the adjoint metric coincides with the original semireducible metric, i.e. if the slice(s) are 1-dimensional. Now, equation (1a) can be rewritten in the form

(1c) $$dS^2 = \varepsilon dl^2 + l^2 d\sigma_4^2, \quad (\varepsilon = -1),$$

(1d) $$d\sigma_4^2 = dt^2 - \sinh^2(t) d\sigma_3^2,$$

where the line element $d\sigma_3^2$ is again given by (1b). Here, we have simplified the expressions by taking, without any loss of generality, $L = 1$. We may also notice that (1b) represents the line element of a 3D space of constant curvature -1. Equations (1c) and (1d) explicitly demonstrate the *semireducible* character of the solution presented by Liko and Wesson and the type of the solution. In particular, using the terminology of section 2, in reference [**3**], we may notice that (1c) is of the type $\{1,4\}$ and that the 4D slice $d\sigma_4^2$ involved, is again a semireducible space of the type $\{1,3\}$. Thus, this specific solution can be interpreted in the context of the semireducible spaces as either a $\{1,\{1,3\}\}$ type of a 5D solution or as a solution of the degenerate type $\{2,2\}$. Further to this point it can be immediately recognized that the line element (1c) is that of a $V(0)$-space, independently of the form of the line element $d\sigma_4^2$. In order to prove our assertion, it is sufficient to write down the *adjoint* metric[2], or more precisely the 2D adjoint line element, of the line element (1c), which is of the form

(2) $$dS_*^2 = \varepsilon dl^2 + l^2 dy^2.$$

Obviously, by inspection this is the line element corresponding to a flat two dimensional space. Hence, the line element (1c) is that of a $V(0)$-pace, by construction and the analysis of reference [**3**] fully applies.

§ 2. **Proof of the statement that (1) is Ricci flat.** In order to prove *analytically* our assertion, it is sufficient to recall theorem 3 and theorem 4 of reference [**3**]. In particular, theorem 3 of the previously mentioned reference states that in order that the $V(0)$-space (1c) is Ricci flat (which is equivalent to saying that (1c) is the line element of a *special Einstein*[3] space), it is necessary and sufficient that the 4D slice is

2) Our terminology is that used in [5] and [7] by Kruckovic and Solodovnikov, a terminology that has been fully adopted in the references [1], [2] and [3].

3) An Einstein space is a space for which, $R_{ij} = \kappa g_{ij}$, where κ is necessarily a constant. In a special Einstein space $\kappa = 0$. The sign and the value of $\kappa = 0$, depends on the conventions adopted.

an Einstein space of the scalar curvature $12K_1$, where K_1 is the conjugate curvature of the slice $d\sigma_4^2$ in the 0-analysis (1c). The proof of this theorem can be found in references [**1**] and [**7**]. Its origin in a disguised form goes back to reference [**8**] and a summary form of the full theory can be found in [**2**]. In our case, $K_1 = -1$. The calculation here is trivial and can be carried out by inspection, by means of equation (2.3) of reference [**3**] (which is the general formula for the calculation of the conjugate curvature in any K-analysis). Now, our task has been simplified considerably. In fact, we may notice that (1d) is a $V(-1)$-space for any possible choice of the slice $d\sigma_3^2$. In order to prove our statement it is sufficient to absorb the minus sign in the definition of the line element $d\sigma_3^2$, which could be negative definite, if necessary. Then the resulting adjoint element is of the form

$$d\sigma_*^2 = dt^2 + \sinh^2(t)dz^2. \tag{3}$$

Clearly, this is the line element of a 2D space of the constant curvature -1 (see equation 6ii of page 272 in reference [4]). Hence, the original semireducible space (1d) is by construction a $V(-1)$-space. However, because of the special choice (1b) i.e. because the slice $d\sigma_3^2$ is that of the constant curvature -1, we can prove that the aforementioned $V(-1)$-space is a space of constant curvature -1. In fact, this is an immediate consequence of theorem 2 of reference [**3**]. In particular, we have that the conjugate curvature of the slice $d\sigma_3^2$ is $+1$, where again the result is obtained by means of equation (2.3) of reference [**3**]. On the other hand the slice $-d\sigma_3^2$ is a space of constant curvature $+1$. Hence, according to the aforementioned theorem 2 (1d) represents a degenerate $V(-1)$-space, i.e. a space of the constant curvature -1. Yet spaces of constant curvature are trivially Einstein spaces. Thus, according to theorem 3, the original $V(0)$-space (1c) is a (special) Einstein space, or in the terminology of Liko and Wesson a Ricci flat space.

§ 3. **Proof of the flatness of the solution (1).** The flatness of the solution (1) is an immediate consequence of our previous proof or alternatively an immediate consequence of theorem 2 of reference [**3**]. In particular, using theorem 3 previously, we have considered the slice $d\sigma_4^2$, which is a degenerate Einstein space, i.e. a space of constant curvature. This means that, in agreement with theorem 2, the resulting $V(0)$-space is also a degenerate special Einstein space, i.e. a locally flat space.

§ 4. **Discussion.** The methodology illustrated above can be used in principle for the construction of arbitrary non flat solutions of the 5D gravity of the type $\{1,4\}$. In particular, it is sufficient to start with any Einstein space of scalar curvature 12ε (the particular value depends on the conventions adopted) and use this space as a slice in (1c). Theorem 3, in reference [**3**], guarantees that the resulting line element represents an *exact* solution of the 5D gravity. It is also worth noticing that the sign of the extra dimension ε, determines the "quality" of the solution since it determines the scalar curvature of the original 4D space which *expands* into the final 5D solution. Now, theorem 2, in reference [**3**] can be used as a criterion for the *flatness* or not of the obtained solution. Further to this point, it has been proved in reference [**3**] that for $n = 5$, there are *eleven* distinct classes of semireducible solutions. Among these types only the type $\{1,4\}$ involves solutions that are $V(0)$-spaces (e.g. the solution of McManus, mentioned by Liko and Wesson in [**6**], is again of the type $\{1,4\}$). This is important because in this case we have all the solutions that are, roughly speaking, closer to the trivial 5D flat vacuum. Clearly, these solutions are "different" from the solutions that are distributed in the remaining *ten* classes and this has been stressed in detail in the aforementioned reference. This property of the solutions of the type $\{1,4\}$, in physical terms, reflects into the fact that the induced 4D energy momentum tensor prescribes the signature of the 4D vacuum.

Section of Astrophysics
Astronomy and Mechanics
Department of Physics
University of Athens
GR 157 84 Zografos, Athens Greece
E-mail: anthkritikos@ath.forthnet.gr

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